Name: ________________________

Math Adventures
Week 7: To Infinity and Beyond!

If something is **infinite**, it is endless.

- **Infinity** is the idea of something that is endless. Infinity is not a number.
- The symbol for infinity is ∞.

In the 1920s, a German mathematician named David Hilbert came up with a paradox about an infinite hotel to show that infinity is very counterintuitive.

Infinite Hotel Rules:

- The hotel has one floor with an infinite number of rooms, numbered (1, 2, 3, ...).
- Only one guest can stay in a room at any given time.
- The guests must change rooms whenever they are asked to do so.

1. The Infinite Hotel is filled with an infinite number of guests (G1, G2, G3, ...), and every room is occupied.

Room 1	Room 2	Room 3	Room 4	Room 5	Room 6	Room 7	Room 8	...
G1	G2	G3	G4	G5	G6	G7	G8	...

   a. One day, a new guest arrives. If we place the new guest (NG) in room 1, how do we relocate the other guests? Fill in the new diagram of the hotel below.

Room 1	Room 2	Room 3	Room 4	Room 5	Room 6	Room 7	Room 8	...
NG								...

   b. If we place the new guest (NG) in room 5, how do we relocate the other guests? Fill in the new diagram of the hotel below.

Room 1	Room 2	Room 3	Room 4	Room 5	Room 6	Room 7	Room 8	...
				NG				...

2. The next day, three new guests arrive. Place the three new guests (NG1, NG2, NG3) and relocate the other guests (G1, G2, G3, ...) in the hotel diagram below.

Room 1	Room 2	Room 3	Room 4	Room 5	Room 6	Room 7	Room 8	...
								...

3. A week later, the Infinite Train arrives at the hotel, bringing an infinite number of new guests.
   a. Can we make room for all the new guests?

   b. If so, place the new guests (NG1, NG2, NG3, ...) and relocate the other guests (G1, G2, G3, ...) in the hotel diagram below.

Room 1	Room 2	Room 3	Room 4	Room 5	Room 6	Room 7	Room 8	...
								...

4. The people in Even Town only use even numbers, and the people in Odd Town only use odd numbers. One day, the Infinite Even Train of Even Town travels to Odd Town, filled with an infinite number of passengers (P2, P4, P6, ...).

Seat 2	Seat 4	Seat 6	Seat 8	Seat 10	Seat 12	Seat 14	...
P2	P4	P6	P8	P10	P12	P14	...

When the train arrives in Odd Town, the passengers decide to stay in the Infinite Odd Hotel. Place the passengers (P2, P4, P6, ...) in the hotel diagram below so that every hotel room is taken.

Room 1	Room 3	Room 5	Room 7	Room 9	Room 11	Room 13	...
							...

Two sets of numbers are the same size if we can pair each number from one set with one number from the other set, so that no numbers are left over in either set. This is also true for infinite sets!

5. Can we pair each odd number with one even number so that no odd numbers nor even numbers are left over? If so, pair the odd numbers with the even numbers below!

Odd	1	3	5	7	9	11	13	15	...
	↕	↕	↕	↕	↕	↕	↕	↕	
Even									...

6. Is the number of even numbers greater than, less than, or equal to the number of odd numbers?

7. After a week vacationing in Odd Town, the guests from Even Town get back into the Infinite Even Train in the same order as before, and the train travels to the original Infinite Hotel that uses both odd and even numbers. Place the passengers (P2, P4, P6, ...) in the hotel diagram below so that every hotel room is taken.

Room 1	Room 2	Room 3	Room 4	Room 5	Room 6	Room 7	Room 8	...
								...

A **natural number** is a number used to count things.

- The natural numbers are 1, 2, 3, 4, ...
- Natural numbers include all the even numbers and all the odd numbers.

8. Can we pair each natural number with one even number so that no natural numbers nor even numbers are left over? If so, pair the natural numbers with the even numbers below!

Natural	1	2	3	4	5	6	7	8	...
	↕	↕	↕	↕	↕	↕	↕	↕	
Even									...

9. Is the number of even numbers greater than, less than, or equal to the number of natural numbers?

10. The newly built Infinite Hotel Deluxe has two floors, each with an infinite number of rooms. Since the hotel was just finished yesterday, the rooms don't have room numbers yet. We know that room 1 should be the first room on the first floor. Starting at room 1, draw a path from room to room to show one possible way of assigning the rest of the room numbers in order. Make sure that your path is a pattern that can be repeated beyond the rooms on this page to eventually cover an infinite number of rooms.

								...
Room 1								...

11. The owners of the Infinite Hotels are planning to build an Infinite Resort that has an infinite number of floors, each with an infinite number of rooms. Although they have drawn the plans for the resort, they have not numbered the rooms yet. We know that room 1 should be the first room on the first floor. Starting at room 1, draw a path from room to room to show one possible way of assigning the rest of the room numbers in order. Make sure that your path is a pattern that can be repeated beyond the rooms on this page to eventually cover an infinite number of rooms.

⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	
								...
								...
								...
								...
								...
								...
								...
Room 1								...

A **rational number** is a number that can be expressed as a fraction.

- For example, 1, ½, ¾, 6, and ⅞ are all rational numbers.
- Rational numbers include all the natural numbers.

12. Below is a grid that lists all of the rational numbers. Starting at 1, can you draw a path from number to number that will eventually pass through every rational number? Try it!

1	2	3	4	5	6	...
$1/2$	$2/2$	$3/2$	$4/2$	$5/2$	$6/2$	...
$1/3$	$2/3$	$3/3$	$4/3$	$5/3$	$6/3$	...
$1/4$	$2/4$	$3/4$	$4/4$	$5/4$	$6/4$	...
$1/5$	$2/5$	$3/5$	$4/5$	$5/5$	$6/5$	...
⋮	⋮	⋮	⋮	⋮	⋮	

13. Can we pair each natural number with one rational number so that no natural numbers nor rational numbers are left over? If so, pair the natural numbers with the rational numbers below!

Natural	1	2	3	4	5	6	7	8	...
	↕	↕	↕	↕	↕	↕	↕	↕	
Rational									...

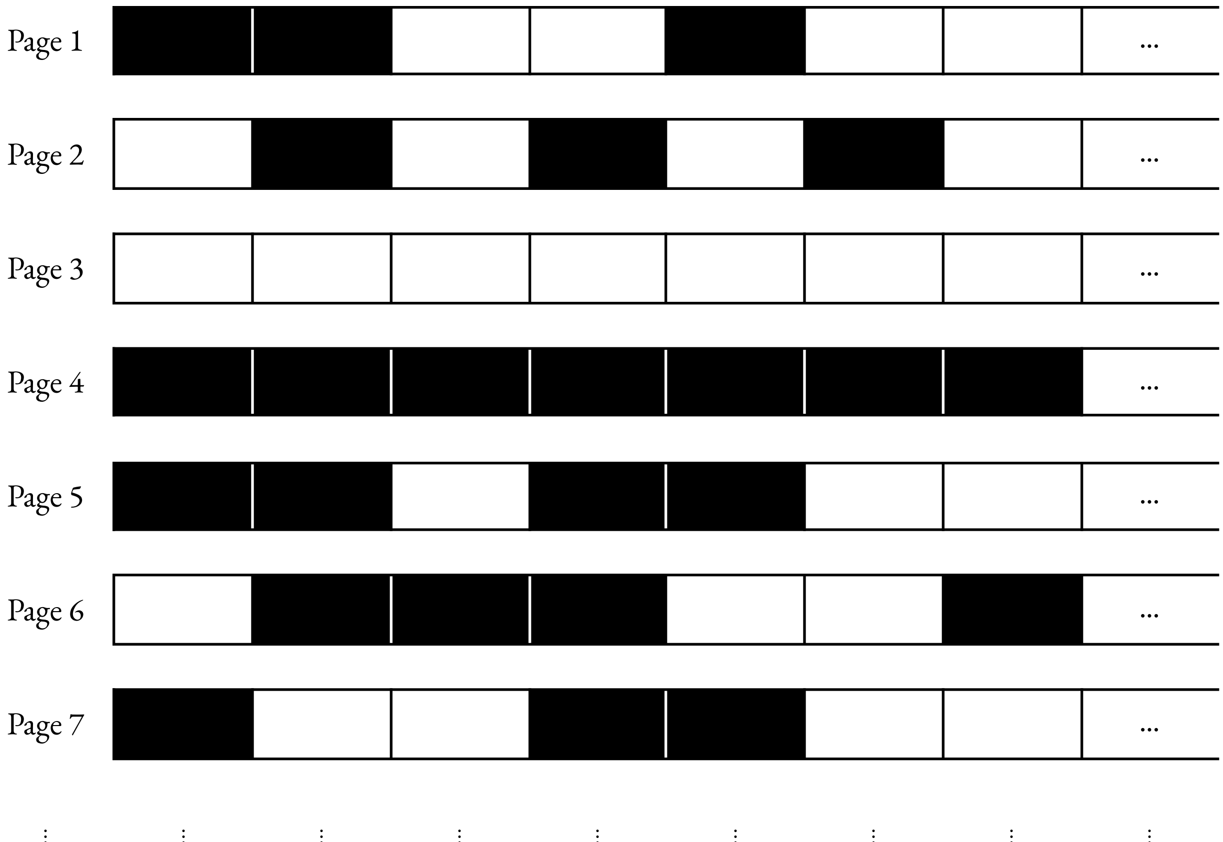
14. Is the number of rational numbers greater than, less than, or equal to the number of natural numbers?

15. At night, some guests at the Infinite Hotel like to sleep with night lights on, while others like to sleep in complete darkness. From the outside, the sequence of light and dark rooms of the hotel looks very pretty. Here is what the hotel looked like last night:

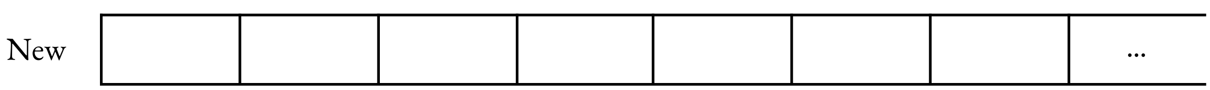
Room 1 Room 2 Room 3 Room 4 Room 5 Room 6 Room 7 Room 8 ...

To help advertise the Infinite Hotel, the marketing team of the hotel wants to create a brochure with an infinite number of pages, numbered 1, 2, 3, ... On each page, the team wants to put a different picture of the hotel at night, so that each page shows off a different sequence of light and dark rooms.

One day, a member of the marketing team named Rotnac announces that he has designed an infinitely long brochure that includes all possible sequences of light and dark rooms. Here are the first few pages of Rotnac's brochure:



a. However, even though Rotnac's brochure shows an infinite number of sequences, some members of the marketing team are skeptical of Rotnac's work and don't believe that he has included *all* possible sequences. You decide to help prove Rotnac wrong. Try to find a sequence that you know cannot be in Rotnac's infinitely long brochure. (Hint: In the new sequence, make room 1 different from room 1 on Rotnac's page 1, room 2 different from room 2 on Rotnac's page 2, room 3 different from room 3 on Rotnac's page 3, etc.)



b. Given *any* infinitely long brochure with an infinite number of sequences, can you always find a sequence that is not already in the brochure?

A **real number** is any number on the number line, even if it cannot be expressed as a fraction.

- For example, 7, π, 1.389575..., 0.25, and 6.666... are all real numbers.
- Real numbers include all the rational numbers.

16. Let's **assume** that we can pair each natural number with one real number between 0 and 1 so that no natural numbers nor real numbers are left over. Suppose our pairing looks like this:

Natural		Real
1	↔	.11612...
2	↔	.50000...
3	↔	.67443...
4	↔	.78737...
5	↔	.23456...
⋮		⋮

Now, let's find a new real number between 0 and 1 that is not already on our list of real numbers! In the new number, make the first decimal place different from the first decimal place of the first real number on our list, the second decimal place different from the second decimal place of the second real number on our list, the third decimal place different from the third real number on our list, etc. Write the first five decimal places of the new number below.

17. Given *any* infinite list of real numbers, can you always find a new real number that is not already on the list?

Therefore, we can never pair each natural number with one real number between 0 and 1 so that no real numbers are left over. **This contradicts our original assumption!**

18. Is the number of real numbers greater than, less than, or equal to the number of natural numbers?

We just wrote a **proof by contradiction**! The discovery that **some infinite sets are larger than other infinite sets** was made in the late 1800s by a German mathematician named Georg Cantor.

Lesson Summary

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- The symbol for infinity is **∞**.

In the 1920s, a German mathematician named David Hilbert came up with a paradox about an infinite hotel to show that infinity is very counterintuitive.

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- For example, 7, π, 1.389575..., 0.25, and 6.666... are all real numbers.
- Real numbers include all the rational numbers.

The sets of even numbers, odd numbers, natural numbers, and rational numbers are all the same size. However, the set of real numbers is larger than these sets! The discovery that **some infinite sets are larger than other infinite sets** was made in the late 1800s by a German mathematician named Georg Cantor.

A **proof by contradiction** is a type of mathematical proof where we make an assumption at the beginning of the proof and then prove that our assumption is incorrect by the end of the proof.

References: Olga Radko Endowed Math Circle archive, *Journey Through Genius* by William Dunham